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Combinatorial basis and non-asymptotic form of the Tsallis entropy function

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Abstract. Using a q-analog of Boltzmann's combinatorial basis of entropy, the non-asymptotic non-degenerate and degenerate combinatorial forms of the Tsallis entropy function are derived. The new measures – supersets of the Tsallis entropy and the non-asymptotic variant of the Shannon entropy – are functions of the probability and degeneracy of each state, the Tsallis parameter q and the number of entities N. The analysis extends the Tsallis entropy concept to systems of small numbers of entities, with implications for the permissible range of q and the role of degeneracy.

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1 Introduction

The combinatorial basis of entropy, given over a century ago by Boltzmann [1] and Planck [2], can be written as [3–6]:

$$H^{(N)} = \frac{1}{N} \ln \mathbb{W} \tag{1}$$

where $H^{(N)}$ is the dimensionless entropy of a system, expressed per unit entity, N is the number of entities and $\mathbb{W}$ is the number of configurations (microstates) corresponding to a specified realization (macrostate) of the system, termed its statistical weight. The maximum entropy position ("MaxEnt") defined by (1), subject to the constraints on the system, therefore corresponds to the realization of maximum weight (hence maximum probability, "MaxProb"), providing a purely probabilistic justification of the entropy concept [1–6]. This perspective is similar to (and subsumes) the "inferential" school of Jaynes [7,8], in which the dimensionless entropy is adopted as a universal tool for statistical inference ("inductive reasoning"), from which the thermodynamic entropy emerges as a special case. However, the combinatorial definition (1) does not depend on information theory. If a system is of multinomial weight:

$$\mathbb{W}_1 = \frac{N!}{\prod_{i=1}^{s} n_i!} \quad \text{with} \quad N = \sum_{i=1}^{s} n_i \tag{2}$$

where n_i is the number of entities in each distinguishable level i, from s such levels, it can be shown using Stirling's approximation $\ln(m!) \approx m \ln m - m$ (or Sanov's theorem [9]) that $H^{(N)}$ converges asymptotically for $N \to \infty, n_i \to \infty, \forall i$ to the Boltzmann-Gibbs-Shannon ("BGS") entropy function [10–12]:

$$H_1^{(\infty)} = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{W}_1 = -\sum_{i=1}^{s} p_i \ln p_i \tag{3}$$

where $p_i = n_i/N$ is the probability of an entity being in the ith level. It must, however, be recognized that a system might not be of multinomial structure. This insight led to the development of other entropy functions, e.g. for Bose-Einstein and Fermi-Dirac statistics, with profound implications for quantum physics [13], and has been explored in the context of other combinatorial structures [14] and non-extensive statistical mechanics [15].

A more recent development in statistical physics has been the proposition of various alternative entropy

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functions, for example the Tsallis entropy [16]:

$$H_q^{(\infty)} = -\sum_{i=1}^{s} p_i^q \ln_q p_i = \frac{1}{q-1}\left(1 - \sum_{i=1}^{s} p_i^q\right) \\ = -\sum_{i=1}^{s} p_i \ln_{2-q} p_i \qquad (4)$$

where $q \in \mathbb{R}$ is the Tsallis parameter, and $\ln_q x = (1-q)^{-1}(x^{1-q}-1), x > 0$ is the q-logarithmic function [17]. In the limit $q \to 1$, $\ln_q x \to \ln x$ and $H_q^{(\infty)}$ reduces to the BGS entropy function (3). The Tsallis entropy has proven useful for the analysis of systems involving long-range interactions, including a diverse range of physical, chemical, astronomical, turbulent, engineering and economic systems (e.g. [18–22]), but to this day remains a topic of some controversy (e.g. [15,23–25]).

To more fully comprehend the physical meaning of the Tsallis entropy, it is important to consider its combinatorial basis. In fact, a combinatorial formulation of the Tsallis entropy – analogous to the Boltzmann principle – has recently been presented, based on a q-logarithmic transformation of a q-multinomial weight, subject to the q-Stirling approximation [26] (these q-algebraic terms are defined in Sect. 2). Embedded within the derivation is the well-known $q \Leftrightarrow (2-q)$ mapping of Tsallis statistics. Separately, the "exact" or "non-asymptotic" forms of the non-degenerate and degenerate BGS entropy functions – valid for finite N and/or $\{n_i\}$ – have also recently been derived, by direct application of Boltzmann's principle without the Stirling approximation [6,27,28]. The latter enables the development of a theory of "non-asymptotic statistical mechanics" (via the method of Jaynes [7,8]) for multinomial systems containing small numbers of entities [27,28]. Parallel analyses, applied to Bose-Einstein and Fermi-Dirac systems, suggest that the "collapse of the wavefunction" in quantum mechanics is a consequence of the (information-theoretic) second law of thermodynamics [27,28].

The aim of this work is to unite the above developments, by examining the non-asymptotic combinatorial forms of the Tsallis entropy function, without and with degenerate levels, as functions of q, $N$and $\{n_i\}$. A Venn diagram for this idea is shown in Figure 1. The derivation makes use of the q-Boltzmann principle [26] and some additional q-algebra, without the q-Stirling approximation. The entropy functions and their corresponding maximum entropy (most probable) distributions are examined. Note that the following analysis is generic, and does not imply any specific connection to thermodynamics, or to any thermodynamic ensemble, except where stated in the text. The approach taken here offers a different perspective to the existing combinatorial analysis of Tsallis statistics, based on the coupling between subsystems of a canonical ensemble (e.g. [29–33]). The analysis has implications for the permissible range of q and role of degeneracy in Tsallis statistics.

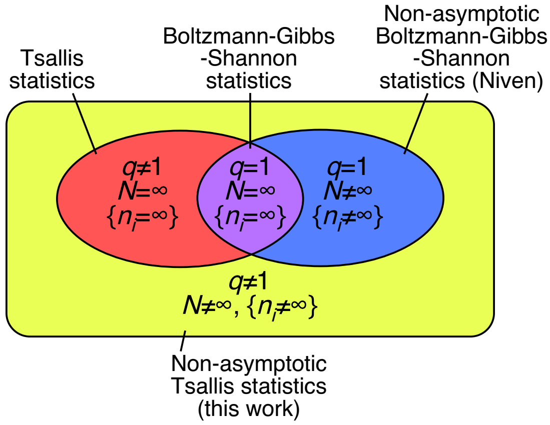

Fig. 1. Venn diagram of the relationships between BGS, Tsallis and non-asymptotic statistical mechanics.

2 Mathematical background

The analysis uses several known and some new q-mathematical functions, as listed below [17,26,34–38]. Unless stated, $q, x, y \in \mathbb{R}$; $a, N, n_i \in \mathbb{N}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; and many functions contain the cutoff condition[1] $[x]_+^{1/(1-q)} = x^{1/(1-q)}\eta_0(x)$, where $\eta_0(x)$ is a modified Heaviside function in which $\eta_0(0) = 0$; hence $[x]_+^{1/(1-q)} = x^{1/(1-q)}$ if $x > 0$ and $[x]_+^{1/(1-q)} = 0$ if $x \leqslant 0$. All functions relate to Tsallis statistics [35–38], and differ from similar terminology used in the mathematics of quantum groups (e.g. [39]).

- q-exponential function [17,35,36]:

$$\exp_q(x) := [1 + (1-q)x]_+^{\frac{1}{1-q}} \qquad (5)$$

- q-logarithm function [17,35,36]:

$$\ln_q x := \frac{x^{1-q}-1}{1-q}, \quad \text{if} \quad x > 0 \qquad (6)$$

- q-product [37,38]:

$$x \otimes_q y := [x^{1-q} + y^{1-q} - 1]_+^{\frac{1}{1-q}}, \quad \text{if} \quad x > 0,\ y > 0 \qquad (7)$$

- q-ratio [37,38]:

$$x \oslash_q y := [x^{1-q} - y^{1-q} + 1]_+^{\frac{1}{1-q}}, \quad \text{if} \quad x > 0,\ y > 0 \qquad (8)$$

- q-power [38]:

$$x^{\wedge}{}_q a = x^{\otimes_q^a} = \underbrace{x \otimes_q x \otimes_q \ldots\ldots \otimes_q x}_{a \text{ times}} \\ = [a x^{1-q} - (a-1)]_+^{\frac{1}{1-q}}, \quad \text{if} \quad x > 0 \qquad (9)$$

where the last form in (9) can be extended to $a \in \mathbb{R}$.

[1] The notation $[x]_+$ has been used incorrectly in Tsallis mathematics, since if $x \leqslant 0$, the resulting zero should not be raised to the power $1/(1-q)$.

- q-factorial [26]:

$$a!_q := 1 \otimes_q \cdots \otimes_q a$$
$$= \left[\left(\sum_{j=1}^{a} j^{1-q} \right) - (a-1) \right]_+^{\frac{1}{1-q}}, \quad \text{if} \quad a > 0 \tag{10}$$

- q-Stirling approximation ("rough" form) [26]:

$$\lim_{a\to\infty} \ln_q(a!_q) = \begin{cases} \frac{a}{2-q}(\ln_q a - 1), & \text{if } q > 0 \text{ and } q \neq 2 \\ a - \ln a, & \text{if } q = 2. \end{cases} \tag{11}$$

For $q = 1$, the traditional Stirling approximation (see Sect. 1) is recovered.

- q-multinomial coefficient [26] – for a system with $N = \sum_{i=1}^{s} n_i$:

$$\mathbb{W}_q = \begin{bmatrix} N \\ n_1 \cdots n_s \end{bmatrix}_q = (N!_q) \oslash_q [(n_1!_q) \otimes_q \cdots \otimes_q (n_s!_q)]$$
$$= \left[(N!_q)^{1-q} + s - \left(\sum_{i=1}^{s} (n_i!_q)^{1-q} \right) \right]_+^{\frac{1}{1-q}},$$
$$\text{if } N > 0, n_i > 0, \forall i \tag{12}$$

whence $\lim_{q\to 1} \mathbb{W}_q = \mathbb{W}_1$ (see (2)). From (6) and (12):

$$\ln_q(\mathbb{W}_q) = \ln_q(N!_q) - \sum_{i=1}^{s} \ln_q(n_i!_q)$$
$$= \frac{1}{1-q} \left\{ (N!_q)^{1-q} - \left[\sum_{i=1}^{s} (n_i!_q)^{1-q} \right] + s - 1 \right\} \tag{13}$$

- q-gamma function [34]:
The q-gamma function $\Gamma_q(x)$, the q-analog of the gamma function $\Gamma(x)$, can be expressed via its q-logarithm as:

$$\ln_q \Gamma_q(x) := \frac{-\zeta(q-1,x) + \zeta(q-1) - x + 1}{1-q}, \quad \begin{cases} x \in \mathbb{R} \\ q \in \mathbb{R} \end{cases} \tag{14}$$

where $\zeta(m,x)$ and $\zeta(m) = \zeta(m,1)$ are the Hurwitz and Riemann zeta functions, respectively. Equation (14) has been proven for $q \in \mathbb{R}, q > 2$, based on an axiomatic definition, and is conjectured for $0 < q < 2$ (the conjecture depends on the analytic continuation of the zeta functions) [34].

- (q,q)-polygamma functions [34]:
The (q,q)-digamma and (q,q) polygamma functions can be defined as, respectively:

$$\Psi_{q,q}(x) = \Psi_{q,q}^{(0)}(x) := \frac{d}{dx} \ln_q \Gamma_q(x) \quad \begin{cases} q \in \mathbb{R} \\ x \in \mathbb{R}, x > 0 \end{cases} \tag{15}$$

$$\Psi_{q,q}^{(m)}(x) := \frac{d^{m+1}}{dx^{m+1}} \ln_q \Gamma_q(x) = \frac{d}{dx} \Psi_{q,q}^{(m-1)}(x), \quad \begin{cases} q \in \mathbb{R} \\ x \in \mathbb{R},\ x > 0 \\ m \in \mathbb{N}. \end{cases} \tag{16}$$

From (14), these give:

$$\Psi_{q,q}(x) = \Psi_{q,q}^{(0)}(x) = -\zeta(q,x) - \frac{1}{1-q}, \quad \begin{cases} q \in \mathbb{R} \\ x \in \mathbb{R},\ x > 0 \end{cases} \tag{17}$$

$$\Psi_{q,q}^{(m)}(x) = \frac{(-1)^{m+1}\Gamma(q+m)\zeta(q+m,x)}{\Gamma(q)}, \quad \begin{cases} q \in \mathbb{R} \\ x \in \mathbb{R},\ x > 0 \\ m \in \mathbb{N}, q+m \neq 1 \end{cases} \tag{18}$$

again proven for $q > 2$ and conjectured for $0 < q < 2$. Note the (q,q)-digamma function differs from the q-digamma function of Yamano [36], in our notation the $(q,1)$-digamma function.

3 Analysis

3.1 Combinatorial basis of entropy

The starting point for the analysis is a generalized form [5,6] of Boltzmann's principle (1):

$$H_{gen} = \kappa(\phi(\mathbb{W}_{gen}) + C) \tag{19}$$

for which:

$$\text{extr } H_{gen} = \sup\ \mathbb{W}_{gen} \tag{20}$$

where H_{gen} is a generalized entropy; $\mathbb{W}_{gen}$ is the statistical weight of any specified realization of a system, of any combinatorial form; $\phi(\cdot)$ is a convenient monotonic transformation function (a generalized or deformed logarithm cf. [40–44]); κ is a scaling parameter; C is an arbitrary constant (reference datum); extr$(\cdot)$ is the extremum and sup$(\cdot)$ the supremum. Equation (20) places a restriction on the choice of κ and ϕ, such that the extremum of H_{gen} recovers the position of maximum $\mathbb{W}_{gen}$, in accordance with the MaxProb principle. Putting $\phi(\cdot) = \ln(\cdot)$, $\kappa = N^{-1}$ and $C = 0$, it can be seen that (19) reduces to the usual Boltzmann principle (1), and thence for a multinomial system, asymptotically to the BGS entropy (3). In general, extremization of H_{gen}, subject to the natural constraint:

$$\sum_{i=1}^{s} p_i = 1 \tag{21}$$

and possibly other constraints, gives the most probable realization of the system.

The purpose of the logarithm in (1) — and of the function $\phi(\cdot)$ in (19) — is to transform the weight into a form which is more easily extremized. If the weight consists of "deformed products" and/or "deformed ratios" of various terms, a convenient choice for $\phi(\cdot)$ is the corresponding deformed logarithm, which can transform this weight into linear sums and/or differences (other desirable properties of deformed logarithms are discussed in [40–44]). For a system of q-multinomial (Tsallis) structure, it is evident that the deformed logarithm $\ln_q(\cdot)$ – or a variant thereof – is an appropriate choice for $\phi(\cdot)$. The factor $\kappa = N^{-1}$ in (1) gives an entropy expressed per unit entity; however, for a q-multinomial system, some other scaling relationship might be expected. Considering (4), (6), (11), (13) and the above arguments, a combinatorial formula for the Tsallis entropy has been found to be [26]:

$$H_q^{(N)} = \frac{q}{N^q} \ln_{2-q}(\mathbb{W}_{2-q}) \quad \text{for} \quad q \neq 0 \tag{22}$$

whence $\lim\limits_{N\to\infty} H_q^{(N)} = H_q^{(\infty)}$. This provides a q-analog of the Boltzmann principle. Note the mathematical $q \Leftrightarrow (2-q)$ mapping between the weight and the entropy (cf. [26]), and also the scaling factor $\kappa = q/N^q$. For $q = 0$, there is no obvious relation between the entropy $H_0 = s-1$ and the reduced weight $\ln_2(\mathbb{W}_2) = -\ln N + \sum_{i=1}^{s} \ln n_i$, suggesting that the Tsallis entropy has no meaning (or a different combinatorial basis) at this point.

3.2 Non-asymptotic combinatorial forms

We now consider the "exact" or "non-asymptotic" combinatorial forms of the above entropy functions, which do not depend on their relevant "deformed Stirling approximation". From the Boltzmann principle (1) and multinomial weight (2), the non-asymptotic BGS entropy function is [27,28]:

$$H^{(N)} = \frac{1}{N} \ln \mathbb{W} = \sum_{i=1}^{s} \left\{ -\frac{1}{N} \ln[(p_i N)!] + \frac{1}{N} p_i \ln[N!] \right\} \tag{23}$$

where, for correct normalization, the $N!$ term is brought inside the summation using the natural constraint (21). Maximization of $H^{(N)}$ subject to (21) and possibly some moment constraints:

$$\sum_{i=1}^{s} p_i f_{ri} = \langle f_r \rangle, \; r = 1, ..., R \tag{24}$$

where f_{ri} is the value of an observable f_r in the ith state and $\langle f_r \rangle$ is its mathematical expectation, yields the non-asymptotic, most probable BGS distribution [27,28], here denoted $p_i^{\#}$:

$$p_i^{\#} = \frac{1}{N} \left[\Psi^{-1} \left(\frac{\ln[N!]}{N} - \lambda_0 - \sum_{r=1}^{R} \lambda_r f_{ri} \right) - 1 \right], \tag{25}$$

where $\psi(x) = y$ is the digamma function and $\psi^{-1}(y)$ is its inverse, in the latter case invoking the uppermost (positive) branch. Note there is no explicit partition function in (25). In the Stirling limits $N \to \infty$ and $\{n_i \to \infty\}, \forall i$, (25) recovers its asymptotic form, denoted p_i^*:

$$p_i^* = \exp\left(-\kappa_0 - \sum_{r=1}^{R} \lambda_r f_{ri} \right) = \frac{1}{Z} \exp\left(-\sum_{r=1}^{R} \lambda_r f_{ri} \right), \quad i = 1, ..., s, \tag{26}$$

where $\kappa_0 = \lambda_0 + 1$ is a shifted first multiplier and $Z = e^{\kappa_0} = \sum_{i=1}^{s} \exp\left(-\sum_{r=1}^{R} \lambda_r f_{ri} \right)$ is the generalized partition function.

Similarly, from the q-Boltzmann principle (22) and q-multinomial weight (12), the non-asymptotic combinatorial form of the Tsallis entropy is obtained as:

$$H_q^{(N)} = \frac{q}{N^q} \ln_{2-q}(\mathbb{W}_{2-q}) = \sum_{i=1}^{s} \left\{ -\frac{q}{N^q} \ln_{2-q}[(p_i N)!_{2-q}] + \frac{q}{N^q} p_i \ln_{2-q}[N!_{2-q}] \right\}, \text{ for } q \neq 0 \tag{27}$$

where the leading $\ln_{2-q}(N!_{2-q})$ term is also brought inside the summation using (21). Equation (27) can be extremized subject to (21) and any number of moment constraints, e.g. of the three different types proposed in Tsallis statistics:
Mark I: Non-power law form, given by (24) [16];
Mark II: Power law form [45]:

$$\sum_{i=1}^{s} p_i^q f_{ri} = \langle f_r \rangle_q, r = 1, ..., R; \tag{28}$$

Mark III: Escort form [46–49]:

$$\frac{\sum\limits_{i=1}^{s} p_i^q f_{ri}}{\sum\limits_{i=1}^{s} p_i^q} = \langle\langle f_r \rangle\rangle_q, r = 1, ..., R, \tag{29}$$

where $\langle f_r \rangle_q$ and $\langle\langle f_r \rangle\rangle_q$ are, respectively, the unescorted and escorted q-expectations of f_r. These give the non-asymptotic, most probable distribution for each case:

$$\text{Mark I:} \quad p_i^{(I)\#} = \frac{1}{N} \left\{ \psi_{2-q,2-q}^{-1} \left[\frac{1}{N} \ln_{2-q}(N!_{2-q}) - \frac{N^{q-1}}{q} \lambda_0^{(I)} - \frac{N^{q-1}}{q} \sum_{r=1}^{R} \lambda_r^{(I)} f_{ri} \right] - 1 \right\}; \tag{30}$$

$$\text{Mark II:} \quad p_i^{(II)\#} = \frac{1}{N} \left\{ \psi_{2-q,2-q}^{-1} \left[\frac{1}{N} \ln_{2-q}(N!_{2-q}) - \frac{N^{q-1}}{q} \lambda_0^{(II)} - \left(N p_i^{(II)\#} \right)^{q-1} \sum_{r=1}^{R} \lambda_r^{(II)} f_{ri} \right] - 1 \right\}; \tag{31}$$

Mark III:

$$p_i^{(III)\#} = \frac{1}{N} \left\{ \psi_{2-q,2-q}^{-1} \left[\frac{1}{N} \ln_{2-q}(N!_{2-q}) - \frac{N^{q-1}}{q} \lambda_0^{(III)} - \frac{\left(N p_i^{(III)\#} \right)^{q-1} \sum\limits_{r=1}^{R} \lambda_r^{(III)} \left(f_{ri} - \langle\langle f_r \rangle\rangle_q \right)}{\sum\limits_{i=1}^{s} \left(p_i^{(III)\#} \right)^q} \right] - 1 \right\} \tag{32}$$

where $\lambda_r^{(j)}, r = 0, ..., R$ are Lagrangian multipliers (different in each case), and $\psi_{q,q}^{-1}(\cdot)$ is the inverse (q,q)-digamma function (see (15), (17)), invoking its uppermost branch. None of the above distributions have an explicit partition function; also, the Mark II and III forms are self-referential. In each case, the above distributions reduce to their recognized forms (e.g. [16,45–49]) (absorbing constant terms into each Lagrangian multiplier $\lambda_0^{(j)}$) when subject to the q-Stirling approximation (11).

As has been noted [27,28], although (25) and (30)–(32) give the most probable distribution of their corresponding system, the variance around this position will be much greater than in the asymptotic case. For very low N and/or n_i (e.g. a few throws of a standard or q-weighted die), it is quite possible that any distribution – not simply the most probable one – will occur. Furthermore, the observed most probable distribution may differ from the predicted distribution (25) or (30)–(32), since the latter may not be realizable due to the effect of quantization [6,27,28], and/or due to the cutoff condition in Tsallis statistics (12). The first two effects are characteristic features of non-asymptotic statistical inference, and indeed of any analysis based on combinatorial arguments (see further discussions in [6,27,28]).

3.3 Plots

To examine the effect of N, several plots of the partial entropy $H_{q,i}^{(N)}$ – defined by the summand of (27) – against p_i are shown in Figure 2, for different values of N and q. In each plot, the discrete values (calculated using q-factorials (10) in (27), quantized with respect to n_i) are shown as points, and their interpolations (calculated using the q-gamma function (14) in (27)) as continuous lines. The calculations were conducted using Maple 9.51, which implements the analytic continuation of the Hurwitz and Riemann zeta functions [50]. Several conclusions can be drawn from these plots:

• For $0 < q < 2$, the partial entropies $H_{q,i}^{(N)}$ are lower than the q-Stirling case $H_{q,i}$, with the dependency on N becoming more spread out as $q \to 0$. From an information-theoretic perspective, this makes sense, since knowledge of N should produce a situation of greater information – and hence lower entropy – than the "default" assumption of infinite N. Accordingly, the non-asymptotic form of the Tsallis entropy $H_q^{(N)}$ is acceptable for $0 < q < 2$.

• For $q = 2$, the partial entropies $H_{q,i}^{(N)}$ coincide with the q-Stirling case $H_{q,i}$, and so there is no dependency on N at this value of q.

• For $q > 2$, the partial entropies $H_{q,i}^{(N)}$ are *higher* than the q-Stirling case $H_{q,i}$. This suggests that knowledge of N imparts *less* information to an observer, which does not make sense from an information-theoretic perspective.

Accordingly, the combinatorial formulation used here appears to suggest that the Tsallis entropy should be restricted to the range $0 < q \leqslant 2$, with specific limits at $q = 1$ and $q = 2$. This result is broadly consistent with similar findings by other authors [18,20,51], using quite different arguments.

4 Degenerate forms

The foregoing analysis may be extended to *degenerate* systems, in which each distinguishable level i contains $g_i \geqslant 1$ distinguishable degenerate sublevels. The sublevels, rather than the levels, are assumed equiprobable. For the multinomial (BGS) case, the weight is:

$$\mathbb{W}_{\mathbf{g}} = N! \prod_{i=1}^{s} \frac{g_i^{n_i}}{n_i!} \tag{33}$$

where $\mathbf{g} = \{g_i\}$. From the Boltzmann principle (1), the non-asymptotic degenerate BGS entropy function is therefore [27,28]:

$$H_{\mathbf{g}}^{(N)} = \sum_{i=1}^{s} \left\{ -\frac{1}{N} \ln[(p_i N)!] + \frac{1}{N} p_i \ln[N!] + p_i \ln g_i \right\} \tag{34}$$

which gives, in the asymptotic limit:

$$H_{\mathbf{g}}^{(\infty)} = \lim_{N\to\infty} H_{\mathbf{g}}^{(N)} = -\sum_{i=1}^{s} p_i \ln \frac{p_i}{g_i}. \tag{35}$$

Similarly, it is possible to define a degenerate q-multinomial coefficient:

$$\begin{aligned} \mathbb{W}_{q,\mathbf{g}} &= (N!_q) \oslash_q [(n_1!_q) \otimes_q \cdots \otimes_q (n_s!_q)] \\ &\quad \otimes_q [(g_1{}^\wedge{}_q n_1) \otimes_q \cdots \otimes_q (g_s{}^\wedge{}_q n_s)] \\ &= \left[(N!_q)^{1-q} - \left(\sum_{i=1}^{s} (n_i!_q)^{1-q} \right) + \left(\sum_{i=1}^{s} n_i g_i^{1-q} \right) \right. \\ &\quad \left. + s - N \right]_+^{\frac{1}{1-q}}, \quad \text{if} \quad N > 0, n_i > 0, \forall i \end{aligned} \tag{36}$$

which satisfies $\lim_{q\to 1} \mathbb{W}_{q,\mathbf{g}} = \mathbb{W}_{\mathbf{g}}$, and from which, using $\ln_q(x \wedge_q a) = a \ln_q x$:

$$\begin{aligned} \ln_q(\mathbb{W}_{q,\mathbf{g}}) &= \ln_q (N!_q) - \sum\nolimits_{i=1}^{s} \ln_q (n_i!_q) + \sum\nolimits_{i=1}^{s} n_i \ln_q (g_i) \\ &= \frac{1}{1-q} \left\{ (N!_q)^{1-q} - \left[\sum_{i=1}^{s} (n_i!_q)^{1-q} \right] \right. \\ &\quad \left. + \left[\sum_{i=1}^{s} n_i g_i^{1-q} \right] + s - N - 1 \right\}. \end{aligned} \tag{37}$$

From (22) and (37), the non-asymptotic, degenerate combinatorial form of the Tsallis entropy function is:

$$\begin{aligned} H_{q,\mathbf{g}}^{(N)} &= \sum_{i=1}^{s} \left\{ -\frac{q}{N^q} \ln_{2-q}[(p_i N)!_{2-q}] + \frac{q}{N^q} p_i \ln_{2-q}[N!_{2-q}] \right. \\ &\quad \left. + \frac{q}{N^{q-1}} p_i \ln_{2-q} g_i \right\}, \quad \text{for} \quad q \neq 0. \end{aligned} \tag{38}$$

In the limit $q \to 1$, this reduces to the non-asymptotic degenerate BGS entropy (34). Alternatively, applying the

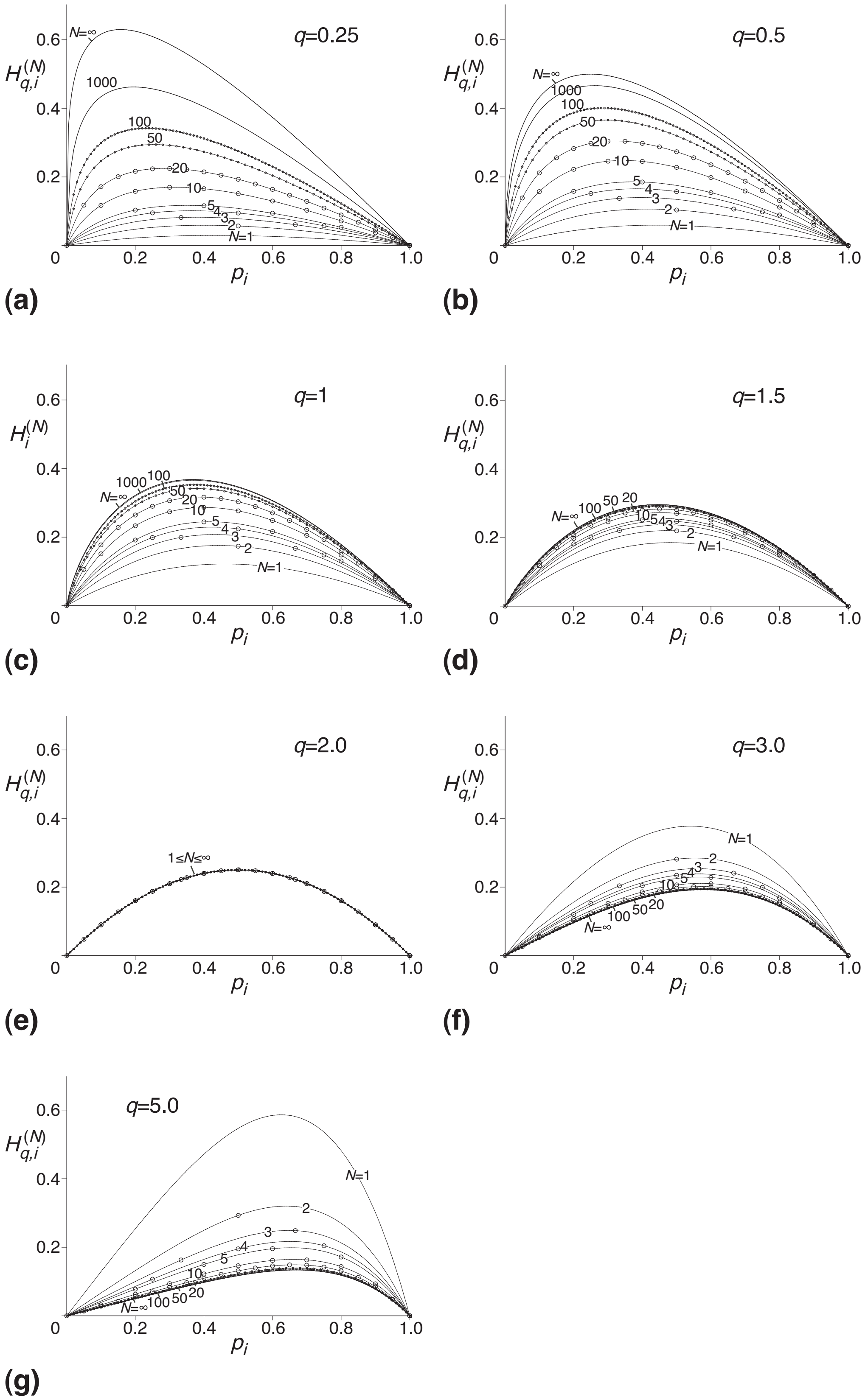

Fig. 2. Plots of the partial non-asymptotic, non-degenerate Tsallis entropy $H_{q.i}^{(N)}$ against p_i , for various values of N and q (points from (10) and (27), and curves from (14) and (27)).

q-Stirling approximation (11) and the relation $\ln_q(p_i N) = N^{1-q}\ln_q p_i + \ln_q N$, (38) reduces to:

$$\begin{aligned} H_{q,\mathbf{g}}^{(\infty)} &= H_{q,\mathbf{g}} \\ &= \lim_{N\to\infty} \sum_{i=1}^{s} \left\{ -p_i \ln_{2-q} p_i + \frac{q}{N^{q-1}} p_i \ln_{2-q} g_i \right\}, \quad \text{for } q \neq 0 \\ &= \begin{cases} H_q, & \text{for} \quad q > 1 \\ H_{\mathbf{g}}, & \text{for} \quad q = 1 \\ H_q + \infty, & \text{for} \quad 0 < q < 1 \\ \text{undefined} & \text{for} \quad q = 0 \\ H_q - \infty, & \text{for} \quad q < 0. \end{cases} \end{aligned} \tag{39}$$

This curious (and unexpected) result indicates that in the limit $N \to \infty$ and $q > 1$, the degenerate Tsallis entropy reduces to the usual (non-degenerate) Tsallis entropy. This result is in stark contrast to that at $N \to \infty$ and $q = 1$ (the degenerate BGS form (35)), in which the degeneracy is of fundamental importance. For $N \to \infty$ and $0 < q < 1$ (or $q < 0$), the degenerate Tsallis entropy is equal to the Tsallis entropy plus (minus) an infinite term. Provided this infinite term can be discarded, then for $N \to \infty$ and $q < 1$ (with $q \neq 0$) the degeneracy again has no effect. The general conclusion is that the Tsallis entropy, for a q-multinomial system in the asymptotic limit, *has no degenerate counterpart*, except for the (BGS) limiting case at $q = 1$.

The maximum-entropy probability distributions and other variants for the non-asymptotic degenerate Tsallis statistic can again be obtained by extremization of (38), subject to the relevant constraint set (21) and (24), (28) or (29).

5 Conclusions

In this work the non-asymptotic combinatorial form of the Tsallis entropy function, , is derived using a q-analog of the combinatorial method of Boltzmann, without the q-Stirling approximation. The new function is a superset of both the Tsallis entropy and the non-asymptotic form of the BGS entropy, containing an additional dependency on N. By information-theoretic reasoning, this formulation provides grounds to suggest that Tsallis statistical mechanics should be restricted to $0 < q \leqslant 2$, with specific limits at $q = 1, 2$. It is also shown that the Tsallis entropy has no degenerate counterpart, except in the specific instances of $q = 1$ or finite N.

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